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Tripartite Probabilistic and Controlled Teleportation of an Arbitrary Single-Qubit State via One-Dimensional Four-Qubit Cluster-Type State

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Abstract A tripartite scheme for probabilistically teleporting an arbitrary single-qubit state with one-dimensional four-qubit cluster-type state as the quantum channel is proposed. In the scheme, both of the sender and the controller perform a Bell-state measurement (BSM) on their respective qubit pair and announce the measurement results via classical communication. With the help of the sender and the controller, the receiver can reconstruct the original state with a certain probability by introducing an auxiliary qubit and making appropriate unitary operations and measurement. In addition, the total success probability and classical message cost of the present scheme are also worked out.

Keywords Controlled teleportation · An arbitrary single-qubit state · One-dimensional four-qubit cluster-type state · Success probability · Classical message cost

1 Introduction

Entanglement is one of the most striking properties in quantum mechanics. As a kind of quantum resource, entanglement has been exploited and applied extensively in the field of quantum information. One of the most remarkable application is quantum teleportation. The first quantum teleportation scheme was proposed by Bennett et al. [1] in 1993. In their scheme, an unknown quantum state can be teleported from a sender to a distant receiver with an Einstein-Podolsky-Rosen (EPR) pair shared as the quantum channel linking the sender and the receiver. The sender first makes a Bell-state measurement (BSM) and publicly announces two-bit classical message corresponding to the measurement result. Then after receiving the sender's classical message, the receiver can reconstruct the original state deterministically by performing a proper unitary operation. Note that no particles carrying the quantum information are needed to send in the process of quantum teleportation.

Since Bennett et al.'s seminal work [1], quantum teleportation has attracted a lot of attention. On one hand, some experimental implementations of quantum teleportation have

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been demonstrated [2–6]. On the other hand, many efforts have devoted to the theoretical extension of quantum teleportation [7–31]. Thereinto, one branch of the extension is controlled teleportation first presented by Karlsson and Bourennane [14] in 1998. In their protocol, an arbitrary single-qubit state can be teleported with a three-qubit Greenberger-Horne-Zeilinger(GHZ) as the quantum channel. The basic idea of controlled teleportation is that the receiver can reconstruct the original state if only getting the help of the sender and the controller. Up to now, a lot of controlled teleportation protocols have been presented [15–31]. For examples, Deng et al. [15] proposed a scheme for teleporting an arbitrary two-qubit state with two GHZ states as the quantum channel. Yan and Wang [16] presented a probabilistic scheme for teleporting arbitrary single- and two-qubit states with three- and four-particle pure entangled state as the quantum channel. Man, Xia and An [17, 18] proposed two deterministic schemes with genuine multi-particle entangled state and EPR pairs plus GHZ states as the quantum channels. Dai, Chen and Li [19, 20] presented two probabilistic protocols with partially three-particle entangled stats as the quantum channel. Gao, Yan and Li [21, 22] proposed two optimal controlled teleportation schemes with general three-particle entangled states as the quantum channel. Moreover, the case of teleporting an arbitrary multi-qubit state has been also investigated [28–31].

Recently, Briegel and Raussendorf [32] introduced a kind of genuine multi-particle entangled state, i.e., one-dimensional cluster state. When $N = 2, 3$, the cluster state is equivalent to Bell state and GHZ state respectively under stochastic local operation and classical communication (LOCC). When $N \geq 4$, they cannot be converted into N-qubit GHZ state and N-qubit W state through LOCC, and they have remarkable properties such as maximal connectedness and persistency of entanglement. Furthermore, the quantum channel linking all the participants is generally a non-maximally entangled pure state because decoherence and noise are inevitable in reality. In this paper we will propose a tripartite scheme for teleporting an arbitrary single-qubit state with a one-dimensional four-qubit cluster-type state as the quantum channel. In the scheme, the sender and the controller both perform a BSM on their respective qubit pair and inform the receiver of their measurement result via classical communication. After receiving the sender's and the controller's classical message, the receiver can reconstruct the original state with a certain success probability by introducing an auxiliary qubit and making proper unitary operations and measurement. Moreover, the total success probability and classical message cost of the present scheme will be worked out.

2 Scheme

Suppose there are three legitimate participants, customarily called Alice, Bob and Charlie, in a quantum network. They are spatially separated in different sites. Now Alice has an arbitrary single-qubit state that initially inhabits in her qubit 1 is in the form

$$|\chi\rangle_1 = \alpha|0\rangle_1 + \beta|1\rangle_1, \tag{1}$$

where α and β are complex and satisfy $|\alpha|^2 + |\beta|^2 = 1$. And she wants to teleport this state to the remote receiver Charlie with the help of Bob. Charlie can reconstruct the state only when he obtains the help of Alice and Bob. Moreover, the quantum channel linking Alice, Bob and Charlie is a one-dimensional four-qubit cluster-type state [32]

$$|\psi\rangle_{2345} = a|0000\rangle_{2345} + b|0011\rangle_{2345} + c|1100\rangle_{2345} - d|1111\rangle_{2345}, \tag{2}$$

where the nonzero real coefficients a, b, c, and d satisfy $a > d$, $b > c$ and $a^2 + b^2 + c^2 + d^2 = 1$. Qubit 2 belongs to Alice, qubits 3 and 4 to Bob and qubit 5 to Charlie. At the time the combined state of these five qubits can be represented as

$$|\Gamma\rangle_{12345} = |\chi\rangle_1 |\psi\rangle_{2345}. \tag{3}$$

The whole scheme can proceed in the following five steps.

Step 1: To achieve her goal, Alice first performs a BSM on her qubit pair $(1, 2)$. After her measurement, the combined state $|\Gamma\rangle_{12345}$ can be written as

$$\begin{aligned}|\Gamma\rangle_{12345} = \frac{1}{\sqrt{2}}[&|\Phi^+\rangle_{12}(\alpha a|000\rangle_{345} + \alpha b|011\rangle_{345} + \beta c|100\rangle_{345} - \beta d|111\rangle_{345})\\ &+ |\Phi^-\rangle_{12}(\alpha a|000\rangle_{345} + \alpha b|011\rangle_{345} - \beta c|100\rangle_{345} + \beta d|111\rangle_{345})\\ &+ |\Psi^+\rangle_{12}(\alpha c|100\rangle_{345} - \alpha d|111\rangle_{345} + \beta a|000\rangle_{345} + \beta b|011\rangle_{345})\\ &+ |\Psi^-\rangle_{12}(\alpha c|100\rangle_{345} - \alpha d|111\rangle_{345} - \beta a|000\rangle_{345} - \beta b|011\rangle_{345})],\end{aligned} \tag{4}$$

where $|\Phi^\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$ and $|\Psi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$ are four Bell states. Then she publishes her measurement results via a classical channel.

Step 2: If the controller Bob agrees to help Charlie to reconstruct the original state, he needs to make a BSM on his qubit pair $(3, 4)$. Afterwards, the whole state $|\Gamma\rangle_{12345}$ can be reexpressed as

$$\begin{aligned}|\Gamma\rangle_{12345} = \frac{1}{2}[&|\Phi^+\rangle_{12}|\Phi^+\rangle_{34}(\alpha a|0\rangle_5 - \beta d|1\rangle_5) + |\Phi^+\rangle_{12}|\Phi^-\rangle_{34}(\alpha a|0\rangle_5 + \beta d|1\rangle_5)\\ &+ |\Phi^+\rangle_{12}|\Psi^+\rangle_{34}(\alpha b|1\rangle_5 + \beta c|0\rangle_5) + |\Phi^+\rangle_{12}|\Psi^-\rangle_{34}(\alpha b|1\rangle_5 - \beta c|0\rangle_5)\\ &+ |\Phi^-\rangle_{12}|\Phi^+\rangle_{34}(\alpha a|0\rangle_5 + \beta d|1\rangle_5) + |\Phi^-\rangle_{12}|\Phi^-\rangle_{34}(\alpha a|0\rangle_5 - \beta d|1\rangle_5)\\ &+ |\Phi^-\rangle_{12}|\Psi^+\rangle_{34}(\alpha b|1\rangle_5 - \beta c|0\rangle_5) + |\Phi^-\rangle_{12}|\Psi^-\rangle_{34}(\alpha b|1\rangle_5 + \beta c|0\rangle_5)\\ &- |\Psi^+\rangle_{12}|\Phi^+\rangle_{34}(\alpha d|1\rangle_5 - \beta a|0\rangle_5) + |\Psi^+\rangle_{12}|\Phi^-\rangle_{34}(\alpha d|1\rangle_5 + \beta a|0\rangle_5)\\ &+ |\Psi^+\rangle_{12}|\Psi^+\rangle_{34}(\alpha c|0\rangle_5 + \beta b|1\rangle_5) - |\Psi^+\rangle_{12}|\Psi^-\rangle_{34}(\alpha c|0\rangle_5 - \beta b|1\rangle_5)\\ &- |\Psi^-\rangle_{12}|\Phi^+\rangle_{34}(\alpha d|1\rangle_5 + \beta a|0\rangle_5) + |\Psi^-\rangle_{12}|\Phi^-\rangle_{34}(\alpha d|1\rangle_5 - \beta a|0\rangle_5)\\ &+ |\Psi^-\rangle_{12}|\Psi^+\rangle_{34}(\alpha c|0\rangle_5 - \beta b|1\rangle_5) - |\Psi^-\rangle_{12}|\Psi^-\rangle_{34}(\alpha c|0\rangle_5 + \beta b|1\rangle_5)].\end{aligned} \tag{5}$$

Then he informs Charlie of his measurement result via a classical channel.

Step 3: Without loss of generality, assume that Alice's and Bob's BSM results are $|\Phi^+\rangle_{12}|\Psi^-\rangle_{34}$ (see (5)), the state of qubit 5 becomes

$$|\phi\rangle_5 = \frac{1}{2}(\alpha b|1\rangle_5 - \beta c|0\rangle_5). \tag{6}$$

In order to reconstruct the original state, Charlie first performs the unitary operation $U_1 = \sigma_5^y = |0\rangle_5\langle 1| - |1\rangle_5\langle 0|$ on qubit 5 to establish a correspondence that the coefficients α and β correspond to $|0\rangle_5$ and $|1\rangle_5$, respectively. After that, the state $|\phi\rangle_5$ is transformed into

$$|\phi'\rangle_5 = \frac{1}{2}(\alpha b|0\rangle_5 + \beta c|1\rangle_5). \tag{7}$$

Table 1 The corresponding relations among Alice's and Bob's BSM results (ABBSMRs), unitary operation U_1, unitary operation $U_2(a_1, a_2)$ and the success probability (SP) for reconstructing the original state. See text for more details

ABBSMRs	U_1	$U_2(a_1, a_2)$	SP
$\|\Phi^+\rangle_{12}\|\Phi^+\rangle_{34}(\|\Phi^-\rangle_{12}\|\Phi^-\rangle_{34})$	σ_5^z	$U_2(1, \frac{d}{a})$	$\frac{d^2}{4}$
$\|\Phi^+\rangle_{12}\|\Phi^-\rangle_{34}(\|\Phi^-\rangle_{12}\|\Phi^+\rangle_{34})$	I_5	$U_2(1, \frac{d}{a})$	$\frac{d^2}{4}$
$\|\Phi^+\rangle_{12}\|\Psi^+\rangle_{34}(\|\Phi^-\rangle_{12}\|\Psi^-\rangle_{34})$	σ_5^x	$U_2(1, \frac{c}{b})$	$\frac{c^2}{4}$
$\|\Phi^+\rangle_{12}\|\Psi^-\rangle_{34}(\|\Phi^-\rangle_{12}\|\Psi^+\rangle_{34})$	σ_5^y	$U_2(1, \frac{c}{b})$	$\frac{c^2}{4}$
$\|\Psi^+\rangle_{12}\|\Phi^+\rangle_{34}(\|\Psi^-\rangle_{12}\|\Phi^-\rangle_{34})$	σ_5^y	$U_2(\frac{d}{a}, 1)$	$\frac{d^2}{4}$
$\|\Psi^+\rangle_{12}\|\Phi^-\rangle_{34}(\|\Psi^-\rangle_{12}\|\Phi^+\rangle_{34})$	σ_5^x	$U_2(\frac{d}{a}, 1)$	$\frac{d^2}{4}$
$\|\Psi^+\rangle_{12}\|\Psi^+\rangle_{34}(\|\Psi^-\rangle_{12}\|\Psi^-\rangle_{34})$	I_5	$U_2(\frac{c}{b}, 1)$	$\frac{c^2}{4}$
$\|\Psi^+\rangle_{12}\|\Psi^-\rangle_{34}(\|\Psi^-\rangle_{12}\|\Psi^+\rangle_{34})$	σ_5^z	$U_2(\frac{c}{b}, 1)$	$\frac{c^2}{4}$

Step 4: Charlie then introduces an auxiliary qubit A with its initial state $|0\rangle_A$ and makes another unitary transformation U_2 on particles 5 and A under the basis $\{|00\rangle_{5A}, |10\rangle_{5A}, |01\rangle_{5A}, |11\rangle_{5A}\}$. The unitary transformation U_2 may take the following 4×4 matrices

$$U_2(a_1, a_2) = \begin{pmatrix} a_1 & 0 & \sqrt{1-a_1^2} & 0 \\ 0 & a_2 & 0 & \sqrt{1-a_2^2} \\ \sqrt{1-a_1^2} & 0 & -a_1 & 0 \\ 0 & \sqrt{1-a_2^2} & 0 & -a_2 \end{pmatrix}, \tag{8}$$

where a_i $(i = 1, 2)$ and $|a_i| < 1$ depends on the states of qubit 5. According to the states of qubit 5 shown in (7), $U_2(a_1, a_2)$ can be chosen as $U_2(\frac{c}{b}, 1)$. Therefore with the transformation $U_2(\frac{c}{b}, 1)$, the state of the qubit 4, 5 and A becomes

$$U_2\left(\frac{c}{b}, 1\right)|\phi'\rangle_5|0\rangle_A = \frac{c}{2}|\chi\rangle_5|0\rangle_A + \frac{1}{2}\alpha\sqrt{b^2-c^2}|01\rangle_{5A}, \tag{9}$$

Step 5: Finally, Charlie measures the auxiliary qubit A in the basis $\{|0\rangle, |1\rangle\}$. Obviously, if his measurement result is $|0\rangle_A$, Charlie knows that he has already reconstructed the original state $|\chi\rangle$ on his qubit 5 with the successful probability $c^2/4$. Otherwise, the scheme fails.

Up to now, we have demonstrated the tripartite probabilistic quantum teleportation scheme for the case of Alice's and Bob's BSM results $|\Phi^+\rangle_{12}|\Psi^-\rangle_{34}$. From (5), We can see that there are 16 possible cases for Alice's and Bob's BSM results. For the other 15 cases, with the same method demonstrated above, the original state can be reconstructed probabilistically in Charlie's site with the help of Alice and Bob. The corresponding relations among Alice's and Bob's BSM results (ABBSMRs), unitary operation U_1 in *Step 3*, unitary operation $U_2(a_1, a_2)$ in *Step 4*, and the success probability for reconstructing the original state is shown in Table 1. Therein, $I = |0\rangle\langle 0| + |1\rangle\langle 1|$, $\sigma_5^x = |0\rangle\langle 1| + |1\rangle\langle 0|$, $\sigma_5^y = |0\rangle\langle 1| - |1\rangle\langle 0|$, and $\sigma_5^z = |0\rangle\langle 0| - |1\rangle\langle 1|$.

From Table 1, we can calculate the total success probability of this scheme

$$P = 2d^2 + 2c^2. \tag{10}$$

In addition, the classical message is an essential resource in quantum teleportation. In *Step 1* and *Step 2*, after performing the BSM, Alice and Bob needs to publicly announce some classical message corresponding to their respective measurement result. The measurement results $|\Phi^{\pm}\rangle_{12}|\Phi^{\pm}\rangle_{34}$, $|\Phi^{\pm}\rangle_{12}|\Psi^{\pm}\rangle_{34}$, $|\Psi^{\pm}\rangle_{12}|\Psi^{\pm}\rangle_{34}$, and $|\Psi^{\pm}\rangle_{12}|\Phi^{\pm}\rangle_{34}$ occur with the probability $p_1 = (\alpha^2 a^2 + \beta^2 d^2)/4$, $p_2 = (\alpha^2 b^2 + \beta^2 c^2)/4$, $p_3 = (\alpha^2 c^2 + \beta^2 b^2)/4$, and $p_4 = (\alpha^2 d^2 + \beta^2 a^2)/4$, respectively. Thus according to the Von Neumann entropy [11], the amount of classical message required in this scheme is

$$\begin{aligned}
\mathcal{S} &= -4\sum_{i=1}^{4} p_i \log_2 p_i \\
&= -(\alpha^2 a^2 + \beta^2 d^2)\log_2(\alpha^2 a^2 + \beta^2 d^2)/4 \\
&\quad - (\alpha^2 b^2 + \beta^2 c^2)\log_2(\alpha^2 b^2 + \beta^2 c^2)/4 \\
&\quad - (\alpha^2 c^2 + \beta^2 b^2)\log_2(\alpha^2 c^2 + \beta^2 b^2)/4 \\
&\quad - (\alpha^2 d^2 + \beta^2 a^2)\log_2(\alpha^2 d^2 + \beta^2 a^2)/4.
\end{aligned} \tag{11}$$

If $|a| = |b| = |c| = |d| = 1/2$, i.e., a one-dimensional four-qubit cluster state is used as the quantum channel, the total probability equals 1. It is obvious that the unitary operations U_1 and U_2 executed by the receiver Charlie are reduced to the unitary operation U_1. In this case, the unitary operation U_2 and the auxiliary qubit A need not be introduced. And the classical message cost is 4 bits. At the time, this probabilistic quantum teleportation scheme is reduced to a deterministic one.

3 Summary

In summary, in this paper we have proposed an explicit tripartite scheme for teleporting an arbitrary single-qubit state with a certain probability. In the scheme, a one-dimensional four-qubit cluster-type state is employed as the quantum channel linking three legitimate participants. The sender Alice performs a BSM on her qubit pair and announces her measurement results publicly via a classical channel. Afterwards, if Bob would like to help Charlie to reconstruct the original state, he needs to makes a BSM on his qubit pair and tells his measurement result to Charlie via a classical channel. Then with Alice's and Bob's classical message, Charlie can reconstruct the original state with a certain success probability by introducing an auxiliary qubit and making appropriate unitary operations and measurement. In addition, the total success probability and classical message of for this scheme are also calculated.

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